Lyapunov stability theory

Alexander Mikhailovich Lyapunov (1857–1918) was a great Russian mathematician. His classic works on the qualitative theory of differential equations, mechanics, mathematical physics, and probability theory are internationally recognized. His theory of stability has countless applications in theoretical and applied research; its influence can hardly be overestimated.

Lyapunov's Stability Theorem. *Recall a fundamental and very elegant theorem on the stability of a stationary solution, to one degree or another familiar to every student of mathematics.*

The motion of a point $y \in \mathbb{R}^n$ with velocity $F(y,t)$, depending on the position of the point and on the time t, is determined by the differential equation

$$y' = F(y,t) \tag{1}$$

and by the position y_0 of the point y at the initial moment t_0.

A solution $y_0(t)$ of equation (1) with the initial data $y_0(t_0) = y_0$ is called *Lyapunov stable* if any solution $y_1(t)$ with sufficiently close initial data y_1 at any moment of time is sufficiently close to $y_0(t)$, that is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|y_1 - y_0| < \delta$ implies $|y_1(t) - y_0(t)| < \varepsilon$ for $t > t_0$, where $y_1(t)$ denotes a solution with the initial data $y_1(t_0) = y_1$.

A solution $y_0(t)$ is called *asymptotically stable* if it is Lyapunov stable and there exists a $\delta > 0$ such that $|y_1 - y_0| < \delta$ implies $\lim_{t\to\infty}(y_1(t) - y_0(t)) = 0$.

Below we consider autonomous differential equations $y' = F(y)$ with $F(0) = 0$, and we discuss the stability problem for the stationary solution $y_0(t) \equiv 0$ of such equations.

Let $\tilde{y}' = A\tilde{y}$ be the linearization of the equation $y' = F(y)$ at the fixed point $y = 0$, where A denotes the differential of the vector function F at 0. Let $\Lambda(A)$ denote the maximum of the real parts for the eigenvalues of the differential A.

Lyapunov's Theorem. *If $\Lambda(A) < 0$, then the stationary solution $y(t) \equiv 0$ to the equation $y' = F(y)$ is asymptotically stable. If $\Lambda(A) > 0$, then the stationary solution is not Lyapunov stable.*

This theorem almost always solves the stability problem for the stationary solution: it fails to apply only in the exceptional case where $\Lambda(A) = 0$.

Lyapunov Theorem splits the spaces of the first degree Taylor polynomials for vector functions F, $F(0) = 0$, into the following three sets:

1) a *stable* set defined by the condition $\Lambda(A) < 0$, for which, regardless of the other coefficients of the Taylor series for the vector function F, the stationary solution is stable;
2) an *unstable set* defined by the condition $\Lambda(A) > 0$, for which, regardless of the other coefficients of the Taylor series for the vector function F, the stationary solution is unstable;
3) a *neutral set* defined by the condition $\Lambda(A) = 0$, for which the stability of the stationary solution depends on the remaining coefficients of the Taylor series for the vector function F.

The partition of the space of 1-jets and vector functions F into stable, unstable, and neutral sets is semi-algebraic: the sign of $\Lambda(A)$ can be determined by calculating the values of a finite number of special polynomials in the coefficients of the differential matrix A.

The relative simplicity of this partition does not mean at all that the question of stability is simple. A similar partition of the space of k-jets of a vector function F into stable, unstable, and neutral sets for a large k is extremely sophisticated, and definitely not semi-algebraic.

Therefore, the very general and relatively simple stability criterion found by Lyapunov should be regarded as a remarkable result and a rare piece of luck. In 1892, he published his fundamental work on the general problem of motion stability. Lyapunov's theory of stability has become classical and is included in the compulsory mathematics program at universities worldwide.

Lyapunov function. Lyapunov has found a surprisingly simple and flexible proof of his stability criterion.

A function $G(y)$ is called a *Lyapunov function* for the dynamical system $y' = F(y)$ if G does not increase as y moves along the trajectory of the dynamical system, that is, if $\frac{dG(y(t))}{dt} = \langle \operatorname{grad} G, F \rangle \leq 0$, where $\operatorname{grad} G$ is the gradient of G and $\langle v, w \rangle$ denotes the scalar product of v and w.

For any constant C, the domain $G \leq C$ is invariant with respect to the dynamical system. Therefore, if the Lyapunov function G has a strict local minimum at y_0, then $y(t) \equiv y_0$ is a Lyapunov stable stationary trajectory of the dynamical system. If, in addition, the strict inequality $\langle \operatorname{grad} G, F \rangle < 0$ is satisfied at the non-critical points of G, then this stationary solution is asymptotically stable.

Indeed, given a differential A with $\Lambda(A) < 0$, it is easy to construct a positive definite quadratic form, which in a neighborhood of the origin is a Lyapunov function of the system under consideration. The existence of such a quadratic form immediately implies the Lyapunov stability of the stationary solution. The rest of the theorem can be verified just as easily.

Imagine a mechanical system, the energy of which is conserved or reduced over time, for example, due to friction. Energy is a Lyapunov function of this system. Damped small oscillations of such a mechanical system near the equilibrium position give a visual representation of the dynamics of the system around the stable equilibrium position described in the Lyapunov Theorem.

Chebyshev's problem. In this part of the note, we use one article by V.A. Steklov, the closest student of A.M. Lyapunov, dedicated to the work of his teacher. As an aspiring mathematician, Lyapunov began solving a problem of P.L. Chebyshev, which he was engaged in until the last days of his life. Chebyshev formulated his problem as follows:

> It is known that a liquid homogeneous mass, whose particles are attracted by Newton's law and which rotates uniformly around a certain axis, can maintain the shape of an ellipsoid as long as the angular velocity ω does not exceed a certain limit.
>
> For values of ω greater than this limit, ellipsoidal figures of equilibrium become impossible.
>
> Let ω be a value of the angular velocity with an equilibrium ellipsoid E. Let us give the angular velocity a sufficiently small increment ε. The question is, are there other equilibrium figures for the angular velocity $\omega+\varepsilon$, not ellipsoidal ones, continuously depending on ε and coinciding with the ellipsoid E for $\varepsilon=0$?

This extremely sophisticated question, connected with the problem of possible forms of celestial bodies, interested many scientists.

Lyapunov obtained the first partial result related to the Chebyshev problem in his master's thesis in 1885.

The great French mathematician A. Poincaré also dealt with this problem. He investigated the first approximation and, on the basis of this approximation to the solution (without rigorous proof and without estimating the error), he came to a conclusion about the existence of an infinite number of different forms of equilibrium close to ellipsoids. He did not know that Lyapunov had reached similar conclusions three years earlier. Poincaré's results were viewed by contemporaries as an outstanding achievement.

In 1901, Lyapunov was elected a full member of the Imperial Academy of Sciences and could devote himself entirely to scientific activities. After that, Lyapunov published a series of memoirs devoted to the Chebyshev problem, which, even from a purely external side, make a strong impression: their volume is over 1000 large-format pages. Using a completely original approach, Lyapunov constructed successive approximations of any order, proved the convergence of the corresponding series and thus obtained a complete solution to the problem.

After the tragic death of A. M. Lyapunov, a completed manuscript of 489 pages remained, containing deep generalizations of his results.

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